A simple systematic method to treat diffusion processes due to given electromagnetic fluctuations

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Background

1. In given electromagnetic fluctuations, there are cases that the transport of test particles is regarded as a diffusion process due to stochastic instability of orbits.
2. In such cases, the diffusions due to electric fluctuations and magnetic fluctuations were considered separately. There is no systematic way to treat such diffusions in a same framework.
3. To develop such a systematic method is quite meaningful even if the fields are given, because the systematic treatment and the results give the most fundamental basis when we interpret the results of self-consistent electromagnetic numerical simulations.

Purpose

1. To develop a simple systematic method to treat the diffusion process of test particles by given **coexisting** electric and magnetic fluctuations.
2. To clarify the differences of the diffusion processes between electric and magnetic fluctuations.
3. To clarify the differences of the diffusion processes between electrons and ions.

Outline

1. A simple systematic method for diffusion processes
2. Application to electromagnetic fluctuations
3. Summary and discussions

1 A simple systematic method for diffusion processes

1. deterministic equation :

   Given fields do not have stochastic properties.

$$\dot{x} = \underbrace{h(x,t)}_{\textbf{part without fluctuation}} + \underbrace{g(x,t)}_{\textbf{part due to fluctuation}}$$

$\Downarrow$ **by stochastic instability of orbits**

2. **Stochastic Differential Equation (SDE)**

   (a) Basic equation

$$\begin{aligned} \dot{x} &= h(x(t),t) + \tilde{g}(x(t),t), \\ \tilde{g}(x(t),t) &= \textbf{stochastic part} \end{aligned}$$

   (b) the formal solution

$$\begin{aligned} x(t) &= x(t; x_0, t_0), \quad x_0 = x(t_0) \\ &= \langle x(t) \rangle + \int_{t_0}^{t} d\tau \tilde{g}(x(\tau),\tau), \\ \langle x(t) \rangle &= x_0 + \int_{t_0}^{t} d\tau h(x_u(\tau),\tau). \\ x_u(t) &= \text{ unperturbed orbit or averaged orbit,} \\ \text{cf} & \quad \text{quasi-linear treatment} \\ x(t) &= \langle x_u(t) \rangle + \int_{t_0}^{t} d\tau \tilde{g}(x_u(\tau),\tau), \end{aligned}$$

   (c) the stochastic properties of $\tilde{g}$:

   Gaussian with no mean value

3. Lagrangian autocorrelation function $\mathcal{R}(t,\tau)$

Let

$$\tilde{g}(x(t),t) \equiv \sum_{k_{||}} \tilde{g}_{k_{||}} \cos(k_{||}x(t) + \delta_{k_{||}} - \omega_{k_{||}}t),$$

then

$$\begin{aligned}
\mathcal{R}(t,\tau) &= \langle \tilde{g}(x(t),t)\tilde{g}(x(\tau),\tau)\rangle \\
&= \left\langle \tilde{g}\left[\langle x(t)\rangle + \int_{t_0}^{t} dt_1 \tilde{g}(x(t_1),t_1),t_1\right]\right. \\
&\times \quad \left.\tilde{g}\left[\langle x(\tau)\rangle + \int_{t_0}^{\tau} dt_2 \tilde{g}(x(t_2),t_2),t_2\right]\right\rangle \\
&= \sum_{k_{||}}\sum_{k'_{||}} \tilde{g}_{k_{||}}\tilde{g}_{k'_{||}} \\
&\times \left\langle \cos\left[k_{||}\langle x(t)\rangle - \omega_{k_{||}}t + \delta_{k_{||}} + k_{||}\int_{t_0}^{t} dt_1 \tilde{g}(x(t_1),t_1)\right]\right. \\
&\quad \left.\cos\left[k'_{||}\langle x(\tau)\rangle - \omega_{k'_{||}}\tau + \delta_{k'_{||}} + k'_{||}\int_{t_0}^{\tau} dt_2 \tilde{g}(x(t_2),t_2)\right]\right\rangle
\end{aligned}$$

Let

$$\begin{aligned}
\alpha(k_{||},t) &\equiv k_{||}\langle x(t)\rangle - \omega_{k_{||}}t + \delta_{k_{||}}, \\
\beta(k_{||},t) &\equiv k_{||}\int_{t_0}^{t} dt_1 \tilde{g}(x(t_1),t_1)
\end{aligned}$$

then (from Gaussian properties)

$$\begin{aligned}
\mathcal{R}(t,\tau) &= \frac{1}{2}\sum_{k_{||}}\sum_{k'_{||}} \tilde{g}_{k_{||}}\tilde{g}_{k'_{||}} \\
&\times \left\{\cos\left[\alpha(k_{||},t) - \alpha(k'_{||},\tau)\right] \exp\left[-\frac{1}{2}\left\langle\left(\beta(k_{||},t) - \beta(k'_{||},\tau)\right)^2\right\rangle\right]\right. \\
&\quad \left.+ \cos\left[\alpha(k_{||},t) + \alpha(k'_{||},\tau)\right] \exp\left[-\frac{1}{2}\left\langle\left(\beta(k_{||},t) + \beta(k'_{||},\tau)\right)^2\right\rangle\right]\right\}
\end{aligned}$$

where

$$
\begin{aligned}
&\left\langle \left(\beta(k_{||}, t) \pm \beta(k'_{||}, \tau) \right)^2 \right\rangle \\
&= k_{||}^2 \int_{\tau}^{t} dt_1 \int_{\tau}^{t} dt_2 \mathcal{R}(t_1, t_2) \\
&+ 2k_{||}(k_{||} \pm k'_{||}) \int_{\tau}^{t} dt_1 \int_{t_0}^{\tau} dt_2 \mathcal{R}(t_1, t_2) \\
&+ (k_{||} \pm k'_{||})^2 \int_{t_0}^{\tau} dt_1 \int_{t_0}^{\tau} dt_2 \mathcal{R}(t_1, t_2)
\end{aligned}
$$

Thus, the Langrangian autocorrelation function is renormalized:

$$
\mathcal{R}(t,\tau) \; = F\left[\mathcal{R}(t,\tau)\right] \;:\; \textbf{Renormalization}
$$

$\langle\rangle$: ensemble average, **We do not use other averages.**

4. Running diffusion coefficient at $x = x_0$

$$
\begin{aligned}
D(t, t_0) &= \frac{1}{2}\frac{d}{dt}\left\langle \left(x(t) - \langle x(t)\rangle\right)^2 \right\rangle, \\
&= \frac{1}{2}\frac{d}{dt}\left\langle \left[\int_{t_0}^{t} d\tau \left[\frac{dx(\tau)}{d\tau} - \left\langle \frac{dx(\tau)}{d\tau} \right\rangle \right] \right]^2 \right\rangle \\
&= \int_{t_0}^{t} d\tau \left\langle \tilde{g}(x(t), t)\tilde{g}(x(\tau), \tau) \right\rangle \\
&= \int_{t_0}^{t} d\tau \mathcal{R}(t, \tau)
\end{aligned}
$$

Long term limit, $\mathcal{R}(t,\tau)$ becomes stationary and has the finite autocorrelation time τ_{ac}:

$$
\begin{aligned}
\mathcal{R}(t,\tau) \quad &\sim \; \mathcal{R}(t-\tau) \;\sim\; \frac{D}{\tau_{ac}} e^{-(t-\tau)/\tau_{ac}} \;:\; \tau_{ac} \sim \frac{1}{k_{||}^2 D} \\
\lim_{t-\tau \gg \tau_{ac}} D(t, t_0) \;&\propto\; \text{const.} \quad : \quad \text{normal diffusion}
\end{aligned}
$$

5. Realization of the stochastic instability

(Based on J.A.Krommes)

$$D \propto \sum_{k_{||}} \tilde{g}^2_{k_{||}} H(x_0, D, k_{||})$$

$$\sum_{k_{||}} = \frac{1}{\Delta k_{||}} \int_{k_{||min}}^{k_{||max}} \text{ : averaging out fine structures}$$

$$D \propto \frac{1}{\Delta k_{||}} \int_{k_{||min}}^{k_{||max}} \tilde{g}^2_{k_{||}} H(x_0, D, k_{||})$$

2 Application to electromagnetic fluctuations

2.1 Deterministic equation

(guiding center approximation based on Little John)

$$\begin{aligned}
\vec{v} &= v_{||}\frac{\vec{B}+\delta\vec{B}+\nabla\times(\rho_{||}\vec{B})}{B+\delta B_{||}+\rho_{||}J_{||}}, \\
&= v_{||}\hat{n}+\vec{v}_{\perp}, \quad \vec{v}_{\perp} \;=\; v_{||}\frac{(\delta\vec{B})_{\perp}+\left[\nabla\times(\rho_{||}\vec{B})\right]_{\perp}}{B+\delta B_{||}+\rho_{||}J_{||}}, \\
\hat{n} &\equiv \frac{\vec{B}}{B}, \quad \rho_{||} \;\equiv\; \frac{v_{||}}{\Omega}, \quad \Omega \;\equiv\; \frac{eB}{m}, \\
\delta\vec{B} &= \nabla\times\delta\vec{A}, \quad \delta\vec{A}=\alpha\vec{B}=\delta A_{||}\hat{n}.
\end{aligned}$$

On $\vec{v}_{\perp}$

$$\begin{aligned}
\text{numerator} &= \frac{\vec{E}\times\vec{B}}{B^2} \;:\; E\times B \text{ drift} \\
&+ \frac{\delta\vec{E}\times\vec{B}}{B^2}+v_{||}\frac{(\delta\vec{B})_{\perp}}{B} \;:\; \text{drifts due to fluctuation} \\
&+ \frac{mv_{\perp}^2}{2eB^3}\vec{B}\times\nabla B+\frac{mv_{||}^2}{eB^4}\vec{B}\times\nabla\left(P+\frac{B^2}{2}\right) \;:\; \text{other drifts} \\
\text{denominator} &= 1+\rho_{||}\frac{J_{||}}{B}+\frac{\delta B_{||}}{B}.
\end{aligned}$$

(Simplification of basic equations)

1. Low-β approximation
2. Large aspect ratio approximation
3. Evaluation of ∇B and curvature drifts based on a model field

$$
\begin{aligned}
B &= B[1 - \varepsilon_t \cos\theta - \varepsilon_h \cos(L\theta - M\zeta)], \\
\varepsilon_t &= \frac{r}{R}, \quad \varepsilon_h = \varepsilon_{ha} \left(\frac{r}{a}\right)^2, \\
& \frac{1}{rB} \left| \left(\frac{r}{R}\right)^2 \iota \frac{\partial B}{\partial \zeta} - \frac{\partial B}{\partial \theta} \right|_{max} \gtrsim \frac{1}{R}, \\
& \frac{1}{B} \left| \frac{\partial B}{\partial r} \right|_{max} \gtrsim \frac{1}{R}.
\end{aligned}
$$

Simplified equations of orbit become

$$
\begin{aligned}
\frac{dr}{dt} &\sim v_{||} \frac{1}{rB} \frac{\partial \delta A_{||}}{\partial \theta} - \frac{1}{rB} \frac{\partial \delta \phi}{\partial \theta} + \frac{\frac{v_\perp^2}{2} + v_{||}^2}{R\Omega}, \\
\frac{d\theta}{dt} &\sim v_{||} \frac{\iota}{R} + \frac{1}{rB} \frac{d\phi}{dr} \\
&- v_{||} \frac{1}{rB} \frac{\partial \delta A_{||}}{\partial r} + \frac{1}{rB} \frac{\partial \delta \phi}{\partial r} + \frac{\frac{v_\perp^2}{2} + v_{||}^2}{R\Omega}, \\
\frac{d\zeta}{dt} &\sim v_{||} \frac{1}{R} - \left(\frac{r}{R}\right)^2 \iota \left\{ \frac{1}{rB} \frac{d\phi}{dr} \right. \\
&\left. - v_{||} \frac{1}{rB} \frac{\partial \delta A_{||}}{\partial r} + \frac{1}{rB} \frac{\partial \delta \phi}{\partial r} + \frac{\frac{v_\perp^2}{2} + v_{||}^2}{R\Omega} \right\}.
\end{aligned}
$$

Only particles satisfying

1. for magnetic fluctuations

$$\left| v_{||} \frac{1}{rB} \frac{\partial \delta A_{||}}{\partial \theta} \right| = \left| v_{||} \frac{\delta B_r}{B} \right| \gg \frac{\frac{1}{2} v_\perp^2 + v_{||}^2}{\Omega R}, \quad \left(\frac{\delta B_r}{B} \gg \frac{\rho}{R} \right),$$

2. for electric fluctuations

$$\left| \frac{1}{rB} \frac{\partial \delta \phi}{\partial \theta} \right| = \left| \frac{\delta E_\theta}{B} \right| \gg \frac{\frac{1}{2} v_\perp^2 + v_{||}^2}{\Omega R}, \quad \left(\frac{\delta E_\theta}{vB} \gg \frac{\rho}{R} \right),$$

mainly contribute to the transport by fluctuated fields.

In opposite limit, the neoclassical transport becomes dominant under the Coulomb collision.

Under the above constraints for the velocity space:
the most simplified equations of orbit are

$$\begin{aligned}
\frac{dr}{dt} &\sim v_{||} \frac{1}{rB} \frac{\partial \delta A_{||}}{\partial \theta} - \frac{1}{rB} \frac{\partial \delta \phi}{\partial \theta}, \\
\frac{d\theta}{dt} &\sim v_{||} \frac{\iota\!\!\!-}{R} - \omega_{E\times B} - v_{||} \frac{1}{rB} \frac{\partial \delta A_{||}}{\partial r} + \frac{1}{rB} \frac{\partial \delta \phi}{\partial r}, \\
\frac{d\zeta}{dt} &\sim v_{||} \frac{1}{R}.
\end{aligned}$$

1. **Only passing particles mainly contribute to the diffusion due to fluctuations**.
2. **the parallel velocity $v_{||}$ is treated as a parameter (not a stochastic variable)**

2.2 Stochastic Differential Equations

1. Simplified deterministic equations:

$$
\begin{aligned}
\frac{dr}{dt} &\sim \quad v_{||}\frac{1}{rB}\frac{\partial \delta A_{||}}{\partial \theta} - \frac{1}{rB}\frac{\partial \delta\phi}{\partial\theta}, \\
\frac{d\theta}{dt} &\sim v_{||}\frac{\iota\!\!\!-}{R} - \omega_{E\times B} \; - v_{||}\frac{1}{rB}\frac{\partial \delta A_{||}}{\partial r} + \frac{1}{rB}\frac{\partial \delta\phi}{\partial r}, \\
\frac{d\zeta}{dt} &\sim v_{||}\frac{1}{R}.
\end{aligned}
$$

2. The form of fluctuations :

$$
\begin{aligned}
\delta A_{||} &= \sum_{mn} \delta A_{||mn}(r)\cos\left[n\zeta - m\theta + \delta_{mn}^{(\delta A)} - \omega_{mn}^{(\delta A)}t\right], \\
\delta\phi &= \sum_{mn} \delta\phi_{mn}(r)\cos\left[n\zeta - m\theta + \delta_{mn}^{(\delta\phi)} - \omega_{mn}^{(\delta\phi)}t\right].
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{g}_r(\vec{r}(t),t) &\equiv v_{||}\frac{1}{rB}\frac{\partial \delta A_{||}}{\partial \theta} - \frac{1}{rB}\frac{\partial \delta\phi}{\partial\theta} \\
&= \sum_{mn}\left\{\frac{v_{||}m}{B}\delta A_{||mn}(r)\sin\left[n\zeta - m\theta + \delta_{mn}^{(\delta A)} - \omega_{mn}^{(\delta A)}t\right]\right. \\
&- \left.\frac{m}{rB}\delta\phi_{mn}(r)\sin\left[n\zeta - m\theta + \delta_{mn}^{(\delta\phi)} - \omega_{mn}^{(\delta\phi)}t\right]\right\} \\
\tilde{g}_\theta(\vec{r}(t),t) &\equiv v_{||}\frac{1}{rB}\frac{\partial \delta A_{||}}{\partial \theta} - \frac{1}{rB}\frac{\partial \delta\phi}{\partial\theta} \\
&= \sum_{mn}\left\{\frac{v_{||}}{B}\frac{\partial \delta A_{||mn}(r)}{\partial r}\cos\left[n\zeta - m\theta + \delta_{mn}^{(\delta A)} - \omega_{mn}^{(\delta A)}t\right]\right. \\
&- \left.\frac{1}{rB}\frac{\partial \delta\phi_{mn}(r)}{\partial r}\cos\left[n\zeta - m\theta + \delta_{mn}^{(\delta\phi)} - \omega_{mn}^{(\delta\phi)}t\right]\right\}
\end{aligned}
$$

3. Corresponding Stochastic Differential Equations:

$$
\begin{aligned}
\frac{dr}{dt} &\sim \tilde{g}_r(\vec{r}(t), t), \\
\frac{d\theta}{dt} &\sim v_{||}\frac{\iota\!\!\!-}{R} - \omega_{E\times B} - \tilde{g}_\theta(\vec{r}(t), t), \\
\frac{d\zeta}{dt} &\sim v_{||}\frac{1}{R},
\end{aligned}
$$

4. The solution of SDE

$$
\begin{aligned}
r(t) &= r(t_0) + \int_{t_0}^{t} d\tau \tilde{g}_r(\vec{r}(\tau), \tau), \\
\theta(t) &= \theta(t_0) + \left[v_{||}\frac{\iota\!\!\!-}{R} - \omega_{E\times B}\right]_{r(t)=r(t_0)} (t - t_0) \\
&+ \left[v_{||}\frac{\iota\!\!\!-'}{R} - \omega'_{E\times B}\right]_{r(t)=r(t_0)} \int_{t_0}^{t} dt' \int_{t_0}^{t'} d\tau \tilde{g}_r(\vec{r}(\tau), \tau) \\
&+ \int_{t_0}^{t} d\tau \tilde{g}_\theta(\vec{r}(\tau), \tau), \\
\zeta(t) &= \zeta(t_0) + v_{||}\frac{1}{R}(t - t_0).
\end{aligned}
$$

2.3 Lagrangian autocorrelation function

$$\begin{aligned}
\mathcal{R}_{rr}(t,\tau) &= \left\langle (r(t)-\langle r(t)\rangle)^2 \right\rangle = \langle \tilde{g}_r(\vec{r}(t),t)\tilde{g}_r(\vec{r}(\tau),\tau)\rangle \\
\mathcal{R}_{r\theta}(t,\tau) &= \langle (r(t)-\langle r(t)\rangle)(\theta(t)-\langle \theta(t)\rangle)\rangle = \langle \tilde{g}_r(\vec{r}(t),t)\tilde{g}_\theta(\vec{r}(\tau),\tau)\rangle \\
\mathcal{R}_{\theta\theta}(t,\tau) &= \left\langle (\theta(t)-\langle \theta(t)\rangle)^2 \right\rangle = \langle \tilde{g}_\theta(\vec{r}(t),t)\tilde{g}_\theta(\vec{r}(\tau),\tau)\rangle
\end{aligned}$$

By using cumulant expansion and Gaussianity with no mean value:

$$\left\langle e^{-ikx}\right\rangle = \exp\left\{\sum_{l=1}^{\infty}\frac{(-ik)^l}{l}C_l\right\} \Rightarrow \left\langle e^{\pm i\xi}\right\rangle = 1 + e^{-\frac{1}{2}\left\langle \xi^2\right\rangle}$$

where

$$\begin{aligned}
\xi &\equiv a\int_{t_0}^{t} dt_1 \tilde{g}_\theta(\vec{r}(t_1),t_1) + b\int_{t_0}^{\tau} dt_2 \tilde{g}_\theta(\vec{r}(t_2),t_2), \\
&\quad + c\int_{t_0}^{t} dt_4\int_{t_0}^{t_4} dt_3 \tilde{g}_r(\vec{r}(t_3),t_3) + d\int_{t_0}^{\tau} dt_6\int_{t_0}^{t_6} dt_5 \tilde{g}_r(\vec{r}(t_5),t_5) \\
\left\langle \xi^2\right\rangle &= a^2\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_1' \mathcal{R}_{\theta\theta}(t_1,t_1') + b^2\int_{t_0}^{\tau} dt_1\int_{t_0}^{\tau} dt_1' \mathcal{R}_{\theta\theta}(t_1,t_1') \\
&+ 2ab\int_{t_0}^{t} dt_1\int_{t_0}^{\tau} dt_1' \mathcal{R}_{\theta\theta}(t_1,t_1') \\
&+ 2ac\int_{t_0}^{t} dt_2\int_{t_0}^{t} dt_1\int_{t_0}^{t_2} dt_1' \mathcal{R}_{r\theta}(t_1,t_1') + 2ad\int_{t_0}^{\tau} dt_2\int_{t_0}^{t} dt_1\int_{t_0}^{t_2} dt_1' \mathcal{R}_{r\theta}(t_1,t_1') \\
&+ 2bc\int_{t_0}^{t} dt_2\int_{t_0}^{\tau} dt_1\int_{t_0}^{t_2} dt_1' \mathcal{R}_{r\theta}(t_1,t_1') + 2bd\int_{t_0}^{\tau} dt_2\int_{t_0}^{\tau} dt_1\int_{t_0}^{t_2} dt_1' \mathcal{R}_{r\theta}(t_1,t_1') \\
&+ c^2\int_{t_0}^{t} dt_2\int_{t_0}^{t} dt_2'\int_{t_0}^{t_2} dt_1\int_{t_0}^{t_2'} dt_1' \mathcal{R}_{rr}(t_1,t_1') \\
&+ d^2\int_{t_0}^{\tau} dt_2\int_{t_0}^{\tau} dt_2'\int_{t_0}^{t_2} dt_1\int_{t_0}^{t_2'} dt_1' \mathcal{R}_{rr}(t_1,t_1') \\
&+ 2cd\int_{t_0}^{t} dt_2\int_{t_0}^{\tau} dt_2'\int_{t_0}^{t_2} dt_1\int_{t_0}^{t_2'} dt_1' \mathcal{R}_{rr}(t_1,t_1')
\end{aligned}$$

$\mathcal{R}_{rr}, \mathcal{R}_{r\theta},$ and $\mathcal{R}_{\theta\theta}$ are functional of $\mathcal{R}_{rr}, \mathcal{R}_{r\theta},$ and $\mathcal{R}_{\theta\theta}$.

$$\mathcal{R}_{rr} = F_{rr}[\mathcal{R}_{\theta\theta},\mathcal{R}_{r\theta},\mathcal{R}_{rr}], \quad \mathcal{R}_{r\theta} = F_{r\theta}[\mathcal{R}_{\theta\theta},\mathcal{R}_{r\theta},\mathcal{R}_{rr}], \quad \mathcal{R}_{\theta\theta} = F_{\theta\theta}[\mathcal{R}_{\theta\theta},\mathcal{R}_{r\theta},\mathcal{R}_{rr}]$$

Long term limit, stationary with an autocorrelation time:

$$\mathcal{R}_{\theta\theta}(t,\tau) \sim \frac{D_{\theta\theta}}{\tau_{ac}^{\theta\theta}}\exp\{-\frac{t-\tau}{\tau_{ac}^{\theta\theta}}\},\ \ \tau_{ac}^{\theta\theta} \propto D_{\theta\theta}^{-1}$$

$$\mathcal{R}_{r\theta}(t,\tau) \sim \frac{D_{r\theta}}{\tau_{ac}^{r\theta}}\exp\{-\frac{t-\tau}{\tau_{ac}^{r\theta}}\},\ \ \tau_{ac}^{r\theta} \propto D_{r\theta}^{-1/2}$$

$$\mathcal{R}_{rr}(t,\tau) \sim \frac{D_{rr}}{\tau_{ac}^{rr}}\exp\{-\frac{t-\tau}{\tau_{ac}^{rr}}\},\ \ \tau_{ac}^{rr} \propto D_{rr}^{-1/3}$$

For $t-\tau \gg \max\{\tau_{ac}^{\theta\theta}, \tau_{ac}^{r\theta}, \tau_{ac}^{rr}\},\ \ t \geq \tau$

$$\begin{aligned}
&\frac{1}{2}\left\langle \xi^2 \right\rangle \\
= &\quad a^2(t-t_0) + (2ab+b^2)(\tau-t_0) && \Leftarrow \mathcal{R}_{\theta\theta} \\
+ &\quad ac(t^2-t_0^2) + (ad+bd)(\tau^2-t_0^2) + 2bc(\tau-t_0)(t-t_0-\frac{1}{2}(\tau-t_0)) && \Leftarrow \mathcal{R}_{r\theta} \\
+ &\quad \frac{1}{2}\{c^2(t-t_0)^3 + (2cd+d^2)(\tau-t_0)^3\} && \Leftarrow \mathcal{R}_{rr}
\end{aligned}$$

Hereafter, we assume

1. Low magnetic shear and low electric shear

$$\iota' \sim 0,\ \ \omega'_{E\times B} \sim 0,$$

because I could not obtain the following indefinite integral:

$$\int dt \sin(\alpha t)e^{-\beta t^3},\ \ \int dt \cos(\alpha t)e^{-\beta t^3},\ \ (\beta > 0)$$

2. for closure

$$m^2\mathcal{R}_{\theta\theta} \sim m^2\left(\frac{\overline{k}_r}{r\overline{k}_\theta}\right)^2\mathcal{R}_{rr} \sim \overline{k}_r^2\mathcal{R}_{rr},\ \ r\overline{k}_\theta \sim \overline{m}$$

$\overline{Q}$: typical value of Q.

2.4 Diffusion coefficient

the radial diffusion coefficient is

$$
\begin{aligned}
D_r &= \lim_{t-t_0 \gg \tau_{ac}} \int_{t_0}^{t} d\tau \mathcal{R}_{rr}(t,\tau) \\
&\sim \frac{1}{2}\sum_{mn} \left\{ \left[v_{||} \frac{m\delta A_{||mn}}{rB} \right]^2 \right. \\
&\qquad \left. - v_{||} \frac{m\delta A_{||mn}}{rB} \frac{m\delta\phi_{mn}}{rB} \cos\left[\delta_{mn}^{(\delta A)} - \delta_{mn}^{(\delta\phi)} - (\omega_{mn}^{(\delta A)} - \omega_{mn}^{(\delta\phi)})(t-t_0) \right] \right\} \\
&\qquad \times \frac{m^2 D_\theta}{\left[k_{||}v_{||} + m\omega_{E\times B} - \omega_{mn}^{(\delta A)} \right]^2 + [m^2 D_\theta]^2} \\
&+ \frac{1}{2}\sum_{mn} \left\{ \left[\frac{m\delta\phi_{mn}}{rB} \right]^2 \right. \\
&\qquad \left. - v_{||} \frac{m\delta A_{||mn}}{rB} \frac{m\delta\phi_{mn}}{rB} \cos\left[\delta_{mn}^{(\delta A)} - \delta_{mn}^{(\delta\phi)} - (\omega_{mn}^{(\delta A)} - \omega_{mn}^{(\delta\phi)})(t-t_0) \right] \right\} \\
&\qquad \times \frac{m^2 D_\theta}{\left[k_{||}v_{||} + m\omega_{E\times B} - \omega_{mn}^{(\delta\phi)} \right]^2 + [m^2 D_\theta]^2} \\
k_{||} &= \frac{n - m\iota\!\!\!-}{R}, \quad \tau_{ac} \sim \frac{1}{\overline{m}^2 D_\theta}
\end{aligned}
$$

Note

$$
m^2 D_\theta \sim m^2 \left(\frac{\overline{k}_r}{r\overline{k}_\theta} \right)^2 D_r \sim \overline{k}_r^2 D_r, \quad r\overline{k}_\theta \sim \overline{m}.
$$

2.5 Realization of the stochastic instability

1. Stochastic instability of orbits is brought by influences of simultaneous multiple waves on orbits.
2. Particles feel infinite number of waves along their perturbed orbits.
3. To express this situation,
   the **finite summation of the discrete** parallel wave number is replaced by the **integration of the continuous** parallel wave number
   information of fine structure of fluctuations is discarded
   $\Rightarrow$ **coarse graining**.

$$\sum_{mn} = \sum_{mk_{||}} \Rightarrow \sum_{m} \frac{1}{\Delta k_{||}} \int_{\delta k_{||min}(<0)}^{\delta k_{||max}(>0)} dk_{||}$$

On $\Delta k_{||}$

$$\langle Q \rangle = \frac{1}{N}\sum_{i=1}^{N} Q_i \sim \frac{\int Q dx}{\int dx} \Rightarrow \sum_{i=1}^{N} Q_i \sim \frac{\int Q dx}{\frac{\int dx}{N}},$$

thus

$$\begin{aligned} \Delta k_{||} &= \frac{\delta k_{||max} - \delta k_{||min}}{N} = \frac{n_{max} - n_{min}}{RN} \sim \frac{1}{R}, \\ L_{||} &= \frac{2\pi}{\Delta k_{||}} \sim 2\pi R \ : \ \textbf{parallel correlation length}. \end{aligned}$$

On $\delta k_{||}$

except for boundaries of the stochastic region,

$$\begin{aligned} -\delta k_{||min} &\sim \delta k_{||max} = \delta k_{||} = \left| \frac{k_\theta}{k_r L_s} \right|, \quad L_s = \frac{R}{\bar{\iota}|s|}, \quad s = \frac{r}{q}\frac{dq}{dr}, \\ \delta k_{||} &\sim \frac{1}{L_s} \ \text{ for } k_\theta \gtrsim k_r, \quad \sim \frac{1}{L_{||}} \ \text{ for } |s| \sim 1. \\ &\sim \left| \frac{k_\theta}{k_r L_s} \right| \ (\gg 1) \text{ for zonal flow with } k_r \gg k_\theta \end{aligned}$$

The resultant diffusion coefficient is

$$
\begin{aligned}
D_r &= \frac{L_{||}}{4\pi}\sum_m \int_{-\delta k_{||}}^{\delta k_{||}} dk_{||} \left\{ \left[v_{||}\frac{m\delta A_{||mk_{||}}}{rB}\right]^2 \right. \\
&\quad \left. - v_{||}\frac{m\delta A_{||mk_{||}}}{rB}\frac{m\delta\phi_{mk_{||}}}{rB}\cos\left[\delta_{mk_{||}}^{(\delta A)} - \delta_{mk_{||}}^{(\delta\phi)} - (\omega_{mn}^{(\delta A)} - \omega_{mn}^{(\delta\phi)})(t-t_0)\right]\right\} \\
&\quad \times \frac{m^2 D_\theta}{\left[k_{||}v_{||} + m\omega_{E\times B} - \omega_{mk_{||}}^{(\delta A)}\right]^2 + [m^2 D_\theta]^2} \\
&+ \frac{L_{||}}{4\pi}\sum_m \int_{-\delta k_{||}}^{\delta k_{||}} dk_{||} \left\{ \left[\frac{m\delta\phi_{mk_{||}}}{rB}\right]^2 \right. \\
&\quad \left. - v_{||}\frac{m\delta A_{||mk_{||}}}{rB}\frac{m\delta\phi_{mk_{||}}}{rB}\cos\left[\delta_{mk_{||}}^{(\delta A)} - \delta_{mk_{||}}^{(\delta\phi)} - (\omega_{mn}^{(\delta A)} - \omega_{mn}^{(\delta\phi)})(t-t_0)\right]\right\} \\
&\quad \times \frac{m^2 D_\theta}{\left[k_{||}v_{||} + m\omega_{E\times B} - \omega_{mk_{||}}^{(\delta\phi)}\right]^2 + [m^2 D_\theta]^2}.
\end{aligned}
$$

By assuming moderate variations of the amplitude and the frequency:

$$
\begin{aligned}
D_r &\sim \frac{L_{||}}{4\pi}\sum_m \left\langle \left[v_{||}\frac{m\delta A_{||mk_{||}}}{rB}\right]^2 \right. \\
&\quad \left. - v_{||}\frac{m\delta A_{||mk_{||}}}{rB}\frac{m\delta\phi_{mk_{||}}}{rB}\cos\left[\delta_{mk_{||}}^{(\delta A)} - \delta_{mk_{||}}^{(\delta\phi)} - (\omega_{mn}^{(\delta A)} - \omega_{mn}^{(\delta\phi)})(t-t_0)\right]\right\rangle_{k_{||}} \\
&\quad \times \int_{-\delta k_{||}}^{\delta k_{||}} dk_{||} \frac{m^2 D_\theta}{\left[k_{||}v_{||} + m\omega_{E\times B} - \left\langle \omega_{mk_{||}}^{(\delta A)}\right\rangle_{k_{||}}\right]^2 + [m^2 D_\theta]^2} \\
&+ \frac{L_{||}}{4\pi}\sum_m \left\langle \left[\frac{m\delta\phi_{mk_{||}}}{rB}\right]^2 \right. \\
&\quad \left. - v_{||}\frac{m\delta A_{||mk_{||}}}{rB}\frac{m\delta\phi_{mk_{||}}}{rB}\cos\left[\delta_{mk_{||}}^{(\delta A)} - \delta_{mk_{||}}^{(\delta\phi)} - (\omega_{mn}^{(\delta A)} - \omega_{mn}^{(\delta\phi)})(t-t_0)\right]\right\rangle_{k_{||}} \\
&\quad \times \int_{-\delta k_{||}}^{\delta k_{||}} dk_{||} \frac{m^2 D_\theta}{\left[k_{||}v_{||} + m\omega_{E\times B} - \left\langle \omega_{mk_{||}}^{(\delta\phi)}\right\rangle_{k_{||}}\right]^2 + [m^2 D_\theta]^2}
\end{aligned}
$$

$$
\begin{aligned}
D_r \;\sim\;& \frac{L_{||}}{4\pi}\sum_m \left\langle v_{||}\left[\frac{m\delta A_{||mk_{||}}}{rB}\right]^2 \right. \\
&\left. - \frac{m\delta A_{||mk_{||}}}{rB}\frac{m\delta\phi_{mk_{||}}}{rB}\cos\left[\delta^{(\delta A)}_{mk_{||}} - \delta^{(\delta\phi)}_{mk_{||}} - (\omega^{(\delta A)}_{mn} - \omega^{(\delta\phi)}_{mn})(t-t_0)\right]\right\rangle_{k_{||}} \\
&\times\left\{ \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} - \left(\left\langle\omega^{(\delta A)}_{mk_{||}}\right\rangle_{k_{||}} - m\omega_{E\times B}\right)}{m^2 D_\theta}\right]\right. \\
&\left. + \;\mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} + \left(\left\langle\omega^{(\delta A)}_{mk_{||}}\right\rangle_{k_{||}} - m\omega_{E\times B}\right)}{m^2 D_\theta}\right]\right\} \\
+\;& \frac{L_{||}}{4\pi}\sum_m \left\langle \frac{1}{v_{||}}\left[\frac{m\delta\phi_{mk_{||}}}{rB}\right]^2 \right. \\
&\left. - \frac{m\delta A_{||mk_{||}}}{rB}\frac{m\delta\phi_{mk_{||}}}{rB}\cos\left[\delta^{(\delta A)}_{mk_{||}} - \delta^{(\delta\phi)}_{mk_{||}} - (\omega^{(\delta A)}_{mn} - \omega^{(\delta\phi)}_{mn})(t-t_0)\right]\right\rangle_{k_{||}} \\
&\times\left\{ \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} - \left(\left\langle\omega^{(\delta\phi)}_{mk_{||}}\right\rangle_{k_{||}} - m\omega_{E\times B}\right)}{m^2 D_\theta}\right]\right. \\
&\left. + \;\mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} + \left(\left\langle\omega^{(\delta\phi)}_{mk_{||}}\right\rangle_{k_{||}} - m\omega_{E\times B}\right)}{m^2 D_\theta}\right]\right\}
\end{aligned}
$$

In the final velocity space integration, the integration of $v_{||}$ may be symmetric, namely, $\left(\int_{-V_{||0}}^{V_{||0}} dv_{||}\right)$, so that the cross term between electric and magnetic fluctuations disappears:

$$\hat{\omega}_m^{(\delta A)} \equiv \left\langle \omega_{mk_{||}}^{(\delta A)} \right\rangle_{k_{||}} - m\omega_{E\times B}, \quad \hat{\omega}_m^{(\delta\phi)} \equiv \left\langle \omega_{mk_{||}}^{(\delta\phi)} \right\rangle_{k_{||}} - m\omega_{E\times B},$$

$$\begin{aligned} D_r(v_{||}, \hat{\omega}_m^{(\delta A)}, \hat{\omega}_m^{(\delta\phi)}) \sim\ & \frac{L_{||}}{4\pi}\sum_m \left\langle v_{||}\left[\frac{m\delta A_{||mk_{||}}}{rB}\right]^2\right\rangle_{k_{||}} \\ & \times\left\{ \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} - \hat{\omega}_m^{(\delta A)}}{m^2 D_\theta}\right] + \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} + \hat{\omega}_m^{(\delta A)}}{m^2 D_\theta}\right]\right\} \\ & + \frac{L_{||}}{4\pi}\sum_m \left\langle \frac{1}{v_{||}}\left[\frac{m\delta \phi_{mk_{||}}}{rB}\right]^2\right\rangle_{k_{||}} \\ & \times\left\{ \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} - \hat{\omega}_m^{(\delta\phi)}}{m^2 D_\theta}\right] + \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} + \hat{\omega}_m^{(\delta\phi)}}{m^2 D_\theta}\right]\right\} \end{aligned}$$

$$\begin{aligned} D_r(v_{||}, \hat{\omega}_m^{(\delta A)}, \hat{\omega}_m^{(\delta\phi)}) \sim\ & \frac{L_{||}}{4\pi}\sum_m \left\langle v_{||}\left[\frac{\delta B_{rmk_{||}}}{B}\right]^2\right\rangle_{k_{||}} \\ & \times\left\{ \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} - \hat{\omega}_m^{(\delta A)}}{m^2 D_\theta}\right] + \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} + \hat{\omega}_m^{(\delta A)}}{m^2 D_\theta}\right]\right\} \\ & + \frac{L_{||}}{4\pi}\sum_m \left\langle \frac{1}{v_{||}}\left[\frac{\delta E_{\theta mk_{||}}}{B}\right]^2\right\rangle_{k_{||}} \\ & \times\left\{ \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} - \hat{\omega}_m^{(\delta\phi)}}{m^2 D_\theta}\right] + \mathrm{Tan}^{-1}\left[\frac{\delta k_{||}v_{||} + \hat{\omega}_m^{(\delta\phi)}}{m^2 D_\theta}\right]\right\} \end{aligned}$$

$$\begin{aligned} D_r(-v_{||}, \hat{\omega}_m^{(\delta A)}, \hat{\omega}_m^{(\delta\phi)}) &= D_r(v_{||}, \hat{\omega}_m^{(\delta A)}, \hat{\omega}_m^{(\delta\phi)}), \\ D_r(v_{||}, -\hat{\omega}_m^{(\delta A)}, -\hat{\omega}_m^{(\delta\phi)}) &= D_r(v_{||}, \hat{\omega}_m^{(\delta A)}, \hat{\omega}_m^{(\delta\phi)}). \end{aligned}$$

In order to obtain limiting cases, $\mathrm{Tan}^{-1}x$ is approximated by

$$\mathrm{Tan}^{-1}x \sim \begin{cases} \dfrac{\pi}{2} & \text{for} \quad x \geq \dfrac{\pi}{2} \\[2ex] x & \text{for} \quad |x| \leq \dfrac{\pi}{2} \\[2ex] -\dfrac{\pi}{2} & \text{for} \quad x \leq -\dfrac{\pi}{2} \end{cases}$$

2.6 Cases with only magnetic fluctuations

Scale separator for magnetic fluctuations : $\mathcal{R}_M$

Δr : radial displacement in the ballistic phase; $\sim v_{rM}\Delta t \sim v_{rM} L_{||}/|v_{||}|$

$L_\perp$: perpendicular correlation length of the fluctuations

$$\mathcal{R}_M \equiv \frac{\Delta r}{L_\perp} = \frac{L_{||}}{L_\perp}\sqrt{\sum_m \left\langle \left(\frac{\delta B_{rmk_{||}}}{B}\right)^2 \right\rangle_{k_{||}}}$$

$\mathcal{R}_M \ll 1 \Rightarrow$ scale separable

Amplitude distinguishes several limits.

1. **Low frequency limit** : $\left|\hat{\omega}_m^{(\delta A)}\right| \ll \frac{\pi}{2}\overline{k}_r^2 D_r$
   (a) **low amplitude limit (quasi-linear limit)**

$$D_r \sim \frac{L_{||}}{4}|v_{||}|\sum_m \left\langle \left(\frac{\delta B_{rmk_{||}}}{B}\right)^2 \right\rangle_{k_{||}}, \text{ for } \overline{k}_r L_\perp \mathcal{R}_M \leq 1.$$

   Averaged (unperturbed) orbits are good approximation.

   (b) **high amplitude limit**

$$D_r \sim |v_{||}|\sqrt{\frac{L_{||}}{2\pi}\sum_m \left\langle \left(\frac{\delta B_{rmk_{||}}}{B}\right)^2 \right\rangle_{k_{||}} \frac{\delta k_{||}}{\overline{k}_r^2}}, \text{ for } \overline{k}_r L_\perp \mathcal{R}_M \geq 1.$$

   Diffusive (perturbed) orbits are good approximation.

2. **High frequency limit** : $\left|\hat{\omega}_m^{(\delta A)}\right| \gg \frac{\pi}{2}\overline{k}_r^2 D_r$
   Averaged (unperturbed) orbits are good approximation.
   (a) **high velocity limit**

$$D_r \sim \frac{L_{||}}{4}|v_{||}|\sum_m \left\langle \left(\frac{\delta B_{rmk_{||}}}{B}\right)^2 \right\rangle_{k_{||}}, \text{ for } |v_{||}| \geq \frac{\left|\hat{\omega}_m^{(\delta A)}\right|}{\delta k_{||}},$$

   (b) **low velocity limit**

$$D_r \sim 0, \text{ for } |v_{||}| < \frac{\left|\hat{\omega}_m^{(\delta A)}\right|}{\delta k_{||}}.$$

2.7 Cases with only electric fluctuations

Scale separator for electric fluctuations : $\mathcal{R}_E$

Δr : radial displacement in the ballistic phase; $\sim v_{rE}\Delta t \sim v_{rE}L_{||}/|v_{||}|$

$L_\perp$: perpendicular correlation length of the fluctuations

$$\mathcal{R}_E \equiv \frac{\Delta r}{L_\perp} = \frac{L_{||}}{L_\perp}\sqrt{\sum_m \left\langle \left(\frac{\delta E_{\theta m k_{||}}}{|v_{||}|B}\right)^2 \right\rangle_{k_{||}}}$$

$\mathcal{R}_E \ll 1 \Rightarrow$ scale separable

Velocity and Amplitude distinguish several limits.

1. **Low frequency limit** : $\left|\hat{\omega}_m^{(\delta\phi)}\right| \ll \frac{\pi}{2}\overline{k}_r^{-2} D_r$
   (a) **high velocity limit**

$$D_r \sim \frac{L_{||}}{4|v_{||}|}\sum_m \left\langle \left(\frac{\delta E_{\theta m k_{||}}}{B}\right)^2 \right\rangle_{k_{||}}, \text{ for } \overline{k}_r L_\perp \mathcal{R}_E \leq 1.$$

Averaged (unperturbed) orbits are good approximation.

   (b) **low velocity limit**

$$D_r \sim \sqrt{\frac{L_{||}}{2\pi}\sum_m \left\langle \left(\frac{\delta E_{\theta m k_{||}}}{B}\right)^2 \right\rangle_{k_{||}} \frac{\delta k_{||}}{\overline{k}_r^2}}, \text{ for } \overline{k}_r L_\perp \mathcal{R}_E \geq 1.$$

Diffusive (perturbed) orbits are good approximation.

2. **High frequency limit** : $\left|\hat{\omega}_m^{(\delta\phi)}\right| \gg \frac{\pi}{2}\overline{k}_r^{-2} D_r$
   Averaged (unperturbed) orbits are good approximation.
   (a) **high velocity limit**

$$D_r \sim \frac{L_{||}}{4|v_{||}|}\sum_m \left\langle \left(\frac{\delta E_{\theta m k_{||}}}{B}\right)^2 \right\rangle_{k_{||}}, \text{ for } |v_{||}| \geq \frac{\left|\hat{\omega}_m^{(\delta\phi)}\right|}{\delta k_{||}},$$

   (b) **low velocity limit**

$$D_r \sim 0, \text{ for } |v_{||}| < \frac{\left|\hat{\omega}_m^{(\delta\phi)}\right|}{\delta k_{||}}.$$

2.8 Cases with electromagnetic fluctuations

1. **low frequency limit** : $\left|\hat{\omega}_m^{(\delta\phi)}\right| \sim \left|\hat{\omega}_m^{(\delta A)}\right| \ll \frac{\pi}{2}\overline{k}_r^2 D_r$

   (a) **low amplitude and high velocity limit**

$$D_r \sim \frac{L_{||}}{4}\sum_m \left\langle |v_{||}| \left[\frac{\delta B_{rmk_{||}}}{B}\right]^2 + \frac{1}{|v_{||}|}\left[\frac{\delta E_{\theta mk_{||}}}{B}\right]^2 \right\rangle_{k_{||}},$$
$$\text{for } \overline{k}_r L_\perp(\mathcal{R}_M + \mathcal{R}_E) \le 1,$$

   Averaged (unperturbed) orbits are good approximation.

   (b) **high amplitude and low velocity limit**

$$D_r \sim \left\{\frac{L_{||}}{2\pi}\sum_m \left\langle \left[|v_{||}|\frac{\delta B_{rmk_{||}}}{B}\right]^2 + \left[\frac{\delta E_{\theta mk_{||}}}{B}\right]^2 \right\rangle_{k_{||}} \frac{\delta k_{||}}{\overline{k}_r^2}\right\}^{1/2},$$
$$\text{for } \overline{k}_r L_\perp(\mathcal{R}_M + \mathcal{R}_E) \ge 1,$$

   Diffusive (perturbed) orbits are good approximation.

2. **high frequency limit** : $\left|\hat{\omega}_m^{(\delta\phi)}\right| \sim \left|\hat{\omega}_m^{(\delta A)}\right| \gg \frac{\pi}{2}\overline{k}_r^2 D_r$

   Averaged (unperturbed) orbits are good approximation.

   (a) **high velocity limit**

$$D_r \sim \frac{L_{||}}{4}\sum_m \left\langle |v_{||}| \left[\frac{\delta B_{rmk_{||}}}{B}\right]^2 + \frac{1}{|v_{||}|}\left[\frac{\delta E_{\theta mk_{||}}}{B}\right]^2 \right\rangle_{k_{||}},$$
$$\text{for } |v_{||}| \ge \frac{\left|\hat{\omega}_m^{(\delta A)}\right|}{\delta k_{||}} \sim \frac{\left|\hat{\omega}_m^{(\delta\phi)}\right|}{\delta k_{||}},$$

   (b) **low velocity limit**

$$D_r \sim 0, \text{ for } |v_{||}| < \frac{\left|\hat{\omega}_m^{(\delta A)}\right|}{\delta k_{||}} \sim \frac{\left|\hat{\omega}_m^{(\delta\phi)}\right|}{\delta k_{||}}.$$

2.9 Velocity space integration

Sorry, I have no time to perform it.
In the case of straight cylindrical tokamak plasmas, there are no constraints on velocity space.

3 Summary and discussions

1. A simple systematic method to treat the diffusion processes is developed on the basis of the Lagrangian autocorrelation function.
2. Analytical expression of the diffusion coefficient is obtained, even if electric and magnetic fluctuations coexist.
3. The differences of diffusion processes between magnetic fluctuations and electric fluctuations will be clarified.
4. The differences of diffusion processes between electrons and of ions will be clarified.

Remaining problems

1. Velocity space integration
2. Extension to finite magnetic shear and $E \times B$ velocity shear
3. Ambipolarity by the radial electric field