Outline of the MHD equilibrium solver in TOPICS

M. Azumi/2006/7/6

Magnetic field and magnetic flux

$$
\begin{aligned}
\mathbf{B} &= \nabla\varphi \times \nabla\psi + T(\psi)\nabla\varphi \\
\Psi &= \int_\psi \mathbf{B}\cdot d\mathbf{S}_p = 2\pi(\psi+\psi_s) \\
\Phi &= \int_\psi \frac{T}{R} dS_\varphi = \frac{\phi(\psi)}{2\pi} \\
\frac{d\Phi}{d\Psi} &= q(\psi) \ , \ \frac{d\psi}{d\phi} = \nu(\psi) = \frac{1}{(2\pi)^2 q(\psi)} \\
\frac{d\phi}{dV} &= T(\psi) < R^{-2} > \\
& V(\psi) = \int_\psi dV \ , \ < f >= \frac{\partial}{\partial V}\int_\psi f dV
\end{aligned}
$$

ψ is defined as $\psi = 0$ at the plasma surface
$\Psi_s = 2\pi\psi_s$ is the surface poloidal flux

G-S equation

$$
\begin{aligned}
\Delta^*\psi(R,Z) &= -RJ(R,Z) \\
&= -R^2(dP/d\psi)(\psi) - (d/d\psi)(T^2/2)(\psi)
\end{aligned}
$$

rationalized MKS unit is used

[1] Initial set up of MHD equilibrium / meudas

Iteration Method of GS-equation solver

$$
\begin{aligned}
\Delta^*\psi_p^{(n)}(R,Z) &= -RJ_p^{(n)}(R,Z) \\
&= -\beta RJ_p[R,\psi^{(n-1)}(R,Z)] - (1-\beta)RJ_p^{(n-1)}(R,Z) \\
-RJ_p[R,\psi] &= -R^2(dP/d\psi)(\psi) - (d/d\psi)(T^2/2)(\psi) \\
\psi^{(n)}(R,Z) &= \psi_p^{(n)}(R,Z) + \psi_s^{(n)} + \sum_k I_{v,k}^{(n)}\psi_{v,k}(R,Z)
\end{aligned}
$$

$dP/d\psi, (d/d\psi)(T^2/2)$: prescrived function of ψ.
$I_{v,k}$: poloidal coil current (fixed or TBD).
$\psi_{v,k}(R,Z)$: vaccume flux function for unit coil current.

$\psi_p^n(R,Z)$ is solved by the Double Cyclic Reduction for the inversion of the operator Δ^* under the boundary condition of

$$\psi_p^{(n)}(\Gamma_b) = \oint_{\Gamma_p} dl_p |\frac{\nabla\psi^{(n-1)}(R_p,Z_p)}{R_p}| G(R_b,Z_b;R_p,Z_p)$$

$\Gamma_b(R_b,Z_b)$: calculation box boundary
$\Gamma_p(R_p,Z_p)$: plasma surface
$G(R,Z;r,z)$: Green Function of GS-equation

Poloidal coil currents are determined so that ψ at prescrived positions minimize the following functional $W(\psi_s, I_{v,k})$:

$$\frac{\partial W}{\partial \psi_s} = 0 \ , \ \frac{\partial W}{\partial I_{v,k}} = 0$$

$$W = \sum_j [\psi_p(R_j,Z_j) + \psi_s^{(n)} + \sum_k I_{v,k}\psi_{v,k}(R_j,Z_j) - \psi_j]^2 + \sum_k w_k [I_{v,k} - I_{v,k}^0]^2$$

$I_{v,k}^0$: prescrived or guessed value of coil current.
w_k : weighting factor.
(R_j,Z_j) : position of marker point (cf. the plasma surface)
ψ_j : prescrived value at the marker point

- As the standard option, marker points (R_j,Z_j) are set on the plasma surface with $\psi_j = 0$.
- For the case like experimental data analysis, marker points (R_j,Z_j) are possible to be set at the flux loop/magnetic probe position with measurered value of ψ_j.
- The plasma surface can be also automatically adjusted to the separatrix for the case of the divertor configuration and to the surface through the limiter for the case of the limiter configulation, depending on the option parameter.

[2] MHD equilibrium during Time Evolution

Definition of radial coordinate in transport equation

$$r = r(\phi)\ ,\ \phi = 2\pi\Phi$$

The definition range of ϕ, ($0 < \phi < \phi_s$) is determined by the initial equilibrium.
Time splitted 2D transport equstions

$$\begin{aligned}
&\frac{d}{dt} n_j(r)V'(r) = 0\\
&\frac{d}{dt} p_j(r)(V'(r))^{\Gamma} = 0\\
&\frac{d}{dt} q(r)^{-1} = 0\\
&\frac{d\Phi(r)}{dt} = \frac{\partial\Phi(r)}{\partial t} + \mathbf{v}\cdot\nabla\Phi(r) = 0
\end{aligned}$$

where $V' = dV/dr$.

Note that the toroidal flux Φ in a plasma is conserved during the plasma evolution and then the definition range of ϕ, or r, is also conserved.

Define

$$\begin{aligned}
\mu &= p_(r)(V'(r))^{\Gamma} = \sum p_j(r)(V'(r))^{\Gamma}\\
\nu &= d\psi/d\phi = (2\pi)^2/q(r)\ ;\ \phi = \Phi/2\pi
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dp}{d\psi} &= \frac{1}{\nu(r)\phi'}\frac{d}{dr}(\frac{\mu(r)}{V'(r)})\\
\frac{d}{d\psi}\frac{T^2}{2} &= \frac{1}{\nu V' < R^{-2} >}\frac{d}{dr}(\frac{\phi'}{< R^{-2} > V'})
\end{aligned}$$

The profile of ψ is evaluated as

$$\psi(r) = \int_r^a \nu(r) d\phi$$

Note that the value of ψ_{axis} at the plasma axis (r=0 ; $\phi = 0$) is determined from this relation, independently from the GS-equation !!

By using the inverse function $r = r(\psi)$, the transport quantities like $\mu(r)$,$\nu(r)$,$dP/d\psi(r)$ etc , are transformed to those in the equilibrium coordinate ψ (or, $V(\psi)$).

$[PDEsolver]$
Considering dP/ψ and $(d/d\psi)(T^2/2)$ as the function of ψ, 2D GS-equation is solved as the partial differential equation by using the same technieque as the initial equilibrium.

$$\Delta^* \bar{\psi}(R, Z) = -R^2 \frac{dP}{d\bar{\psi}}(\bar{\psi}) - \frac{d}{d\bar{\psi}} \frac{T^2}{2}(\bar{\psi})$$

The important point of this solution is that the absolute value of $\bar{\psi}(R, Z)$ does not have any meaning but the 2d-profile $\bar{\psi}(R, Z)$gives the geometrical information of the equilibrium configuration.
That is; introducing the volume V surrounded by $\bar{\psi}$ as $V(\bar{\psi}) = \int_{\bar{\psi}} dV$, the equiribrium configulation is expresssed as

$$V(R, Z) = V(\bar{\psi}(R, Z))$$

and all equilibrium and surface averaged quantities are expressed as the function of V.

$[ODEsolver]$
By averaging the G-S equation along the flux surface, the following surface averaged G-S equation is obtained:

$$\frac{d}{dV}(< |\frac{\nabla V}{R}|^2 > \frac{d\psi}{dV}) = 2\pi \frac{dI}{dV} = -\frac{dP}{d\psi} - < R^{-2} > \frac{d}{d\psi} \frac{T^2}{2}$$

R.H.S also contains $d^2\psi/dV^2$, $d\psi/dV$ and ψ. This equation is the ordinary differential equation for ψ in the coordinate V and is solved for the boundary conditions of $\psi(V = 0) = \psi_{axis}$ and $\psi(V = V_a) = 0$ for given function of $\nu(\phi(V))$, $\mu(\phi(V))$ and surface averaged quantities evaluated from PDE.

ODE gives $\psi(V)$ and, using the relation of $r = r(\psi)$, all equilibrium quantities (metrics, geometrical factors, etc) are transformed to those in transport corordinate.

Finally, the 2 dimentional equilibrium solution of $\psi(R, Z)$ is given by

$$\psi(R, Z) = \psi(V(\bar{\psi}(R, Z)))$$